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Strong Uniqueness and Lipschitz Continuity of Metric Projections: A Generalization of the Classical Haar Theory

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We generalize the concept of strong uniqueness of the metric projection P_G under Hausdorff metric. We show that, under this metric, the following statements are equivalent:

(i) P_G is continuous;

(ii) P_G is pointwise Lipschitz continuous;

(iii) P_G is pointwise strongly unique.

1. INTRODUCTION

Let T be a locally compact Hausdorff space and let $C_0(T)$ be the Banach space of real-valued continuous functions f on T which vanish at infinity, i.e., the set $\{t \in T: |f(t)| \geqslant \varepsilon\}$ is compact for every $\varepsilon > 0$. $C_0(T)$ is endowed with the supremum norm:

$$\|f\| = \sup\{|f(t)|: t \in T\}.$$

For two subsets A, B in $C_0(T)$, define

$$d(A, B) = \sup_{f \in A} \inf_{g \in B} \|f - g\|,$$

$$D(A, B) = \max\{d(A, B), d(B, A)\}.$$

Here $D(A, B)$ is called the Hausdorff metric of A and B. For a finite-dimen-

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sional subspace G of $C_0(T)$, the metric projection P_G from $C_0(T)$ to G is defined as

$$P_G(f = \{g \in G: \|f-g\| = d(f, G)\}, \qquad f \in C_0(T).$$

There are very nice characterizations which ensure the uniqueness of P_G.

THEOREM A. *Suppose that G is a finite-dimensional subspace of $C_0(T)$. Then the following are equivalent*:

(i) *G satisfies the Haar condition*; *i.e., every nonzero $g \in G$ has at most* $\dim G - 1$ *zeros*;

(ii) *$P_G(f)$ is unique (i.e., is a singleton) for all $f \in C_0(T)$*;

(iii) *for every $f \in C_0(T)$, $P_G(f)$ is strongly unique*; *i.e., there exists $r(f) > 0$ such that*

$$\|f-g\| \geqslant d(f, G) + r(f) \cdot \|g - P_G(f)\|, \qquad g \in G;$$

(iv) *for every $f \in C_0(T)$, P_G is Lipschitz continuous at f*; *i.e., there exists $s(f) > 0$ such that*

$$\|P_G(f) - P_G(h)\| \leqslant s(f) \cdot \|f-h\|, \qquad h \in C_0(T).$$

Furthermore, if $T = [a, b]$, then all the above are equivalent to

(v) $U_G = SU_G$,

where $U_G = \{f \in C_0(T)$: $P_G(f)$ is unique$\}$ and $SU_G = \{f \in C_0(T)$: $P_G(f)$ is strong unique$\}$.

The equivalence of (i) and (ii) is proved by Young [16], Haar [9], and Phelps [15]. Freud shows that (i) implies (iv) [7]. That (i) implies (iii) is a result of Newman and Shapiro [12]. The equivalence of (i) and (v) is established by MacLaughlin and Somers [11]. And Cheney proves that (iii) implies (iv) [6]. Now the Lipschitz continuity and strong uniqueness of P_G become an interesting topic in approximation theory (see [1, 2, 3, 13, 14] and references therein).

The main purpose of this paper is to develop an analogous theorem for the multi-valued metric projection P_G. A natural generalization of strong uniqueness for the multi-valued metric projection P_G seems to be the following:

DEFINITION. $P_G(f)$ is called Hausdorff strongly unique if there exists $r(f) > 0$ such that

$$\|f-g\| \geqslant d(f, G) + r(f) \cdot d(g, P_G(f)), \qquad g \in G.$$

Recall that P_G is Hausdorff continuous at f if

$$\lim_{\varepsilon \to 0^+} \sup_{\|h-f\| \leqslant \varepsilon} D(P_G(f), P_G(h)) = 0;$$

and P_G is Hausdorff–Lipschitz continuous at f if there exists $s(f) > 0$ such that

$$D(P_G(f), P_G(h)) \leqslant s(f) \cdot \|f-h\|, \qquad h \in C_0(T).$$

Then the main results of this paper can be summarized as follows:

THEOREM 1. *Suppose that G is a finite-dimensional subspace of $C_0(T)$. Then the following statements are mutually equivalent:*

(i) *for every nonzero $g \in G$,*

$$\operatorname{card}(bdZ(g)) \leqslant \dim\{p \in G\colon \operatorname{int} Z(g) \subset Z(p)\} - 1; \tag{1}$$

(ii) *P_G is Hausdorff continuous at every $f \in C_0(T)$;*

(iii) *$P_G(f)$ is Hausdorff strongly unique for all $f \in C_0(T)$;*

(iv) *P_G is Hausdorff–Lipschitz continuous at every $f \in C_0(T)$.*

If T contains no isolated points, then all above are equivalent to

(v) $U_G = SU_G$.

Furthermore, if T is connected, then all above are equivalent to the fact that G satisfies the Haar condition.

Here $Z(g)$ is the set of all zeros of g and $\operatorname{card}(bdZ(g))$ denotes the cardinal number of the boundary set of $Z(g)$.

By Theorem 1 we know that condition (1) is a natural generalization of the Haar condition and generally $SU_G = U_G$ is not a characteristic description of the Haar condition.

Remark. A nonintrinsic characterization of Hausdorff continuous metric projections was given in [4]. A consequence of this result is that P_G is Hausdorff continuous if and only if G satisfies the Haar condition, provided that T is connected [4]. Moreover, for $T = N$ (i.e., $C_0(T) = c_0$) it was proved in [4] that P_G is Hausdorff continuous and in [14] that $U_G = SU_G$ for an arbitrary finite-dimensional space G of C_0.

2. THE EQUIVALENCE OF HAUSDORFF STRONG UNIQUENESS AND HAUSDORFF–LIPSCHITZ CONTINUITY

From now on, we always assume that G is a finite-dimensional subspace of $C_0(T)$. Since P_G is upper semicontinuous (i.e., for any $f \in C_0(T)$,

$d(P_G(h), P_G(f)) \to 0$ as $h \to f$), P_G is Hausdorff continuous at f if and only if P_G is lower semicontinuous at f (i.e., $d(P_G(f), P_G(h)) \to 0$ as $h \to f$). Our proofs are based on the following theorem:

THEOREM B. *P_G is Hausdorff continuous at f if and only if $E(f-P_G(f)) \subset \operatorname{int}\{t \in T: p(t)-g(t)=0$ for all $p, g \in P_G(f)\}$, where $E(f-P_G(f)) = \{t \in T: |f(t)-g(t)| = d(f, G)$ for all $g \in P_G(f)\}$.*

Theorem B is announced in [5] and can be deduced from the proof of Theorem 2 in [4].

First we show that Hausdorff strong uniqueness is closely related to Hausdorff–Lipschitz continuity.

LEMMA 1. *Suppose that P_G is Hausdorff continuous at $f \in C_0(T) \setminus G$. Then the following statements are mutually equivalent:*

(i) *there exists $r > 0$ such that*

$$\sup\{(f(t)-P_G(f))\, p(t): t \in E(f-P_G(f))\} \geq r \cdot \|p\|_V, \qquad p \in G,$$

where $V = \operatorname{int}\{t \in T: p(t)-g(t)=0$ for all $p, g \in P_G(f)\}$ and $\|p\|_V = \sup\{|p(t)|: t \in V\}$;

(ii) *$P_G(f)$ is Hausdorff strongly unique;*

(iii) *P_G is upper Hausdorff–Lipschitz continuous at f; i.e., there exists $s > 0$ such that*

$$d(P_G(h), P_G(f)) \leq s \cdot \|h-f\|, \qquad h \in C_0(T);$$

(iv) *P_G is Hausdorff–Lipschitz continuous at f.*

Proof. We first show some simple facts. From Theorem B, we have

$$E(f-P_G(f)) \subset V. \tag{2}$$

Set $g^* \in P_G(f)$ such that

$$E(f-g^*) = E(f-P_G(f)) \subset V. \tag{3}$$

Let $\delta = d(f, G) - \max\{|f(t)-g^*(t)|: t \in T \setminus V\}$. Then

$$g^* + p \in P_G(f), \qquad \text{for } p \in G \text{ with } V \subset Z(p) \text{ and } \|p\| \leq \delta. \tag{4}$$

Set $G(f) = \operatorname{span}\{p-g: p, g \in P_G(f)\}$. Then for some $c > 0$,

$$d(p, G(f)) \leq c\, \|p\|_V, \qquad p \in G. \tag{5}$$

In fact, if (5) fails to be true, then for some $p \in G \setminus G(f)$,

$$V \subset Z(p).$$

From (4) we obtain that for some $\lambda > 0$, $g^* + \lambda p \in P_G(f)$, i.e.,

$$p \in G(f)/\lambda = G(f).$$

This is impossible.

Now we begin to investigate the relations among the statements in Lemma 1.

(i) $\Rightarrow$ (ii). By (i) and the strong Kolmogorov criterion [13], we deduce that there exists $r(f) > 0$ such that

$$\|f-p\| \geqslant d(f, G) + r(f) \cdot \|p - P_G(f)\|_V, \qquad p \in G. \tag{6}$$

Assume that statement (ii) is not true; i.e., there exist $p_n \in G \setminus P_G(f)$ such that

$$\|f-p_n\| \leqslant d(f, G) + \frac{1}{n} d(p_n, P_G(f)), \qquad n \geqslant 1. \tag{7}$$

Let $g_n \in P_G(f)$ such that $d(p_n, P_G(f)) = \|p_n - g_n\|$. From (6) and (7) we get

$$\|p_n - g_n\|_V / \|p_n - g_n\| \to 0, \qquad \text{as} \quad n \to \infty. \tag{8}$$

By selecting a subsequence, we may assume

$$(p_n - g_n)/\|p_n - g_n\| \to p, \qquad \text{as} \quad n \to \infty.$$

Equation (8) implies $V \subset Z(p)$. By (5), $p \in G(f)$. Set

$$q_n = g_n + \|p_n - g_n\| \cdot p.$$

Since $\|p_n - q_n\|/d(p_n, P_G(f)) = \|p_n - q_n\|/\|p_n - g_n\| \to 0$ as $n \to \infty$, we obtain that

$$d(q_n, P_G(f))/d(p_n, P_G(f)) \to 1 \qquad \text{as} \quad n \to \infty.$$

Thus we derive from (7) that

$$(\|f - q_n\| - d(f, G))/d(q_n, P_G(f)) \to 0 \qquad \text{as} \quad n \to \infty. \tag{9}$$

But $V \subset Z(p) \cap Z(g_n - g^*) \subset Z(q_n - g^*)$ and $q_n \bar{\in} P_G(f)$. By (4), there exists $1 > \lambda_n > 0$ such that

$$g^* + \lambda_n(q_n - g^*) \in P_G(f), \tag{10}$$

and

$$g^* + \lambda(q_n - g^*) \bar{\in} P_G(f), \qquad \text{for} \quad \lambda > \lambda_n. \tag{11}$$

Set $u_n = g^* + \lambda_n(q_n - g^*)$. From (10) and (11) we obtain

$$\max\{(f(t) - u_n(t))(q_n(t) - g^*(t)) : t \in E(f - u_n) \setminus V\} \geqslant 0. \tag{12}$$

Let $t_n \in E(f - u_n) \setminus V$ such that

$$(f(t_n) - u_n(t_n))(q_n(t_n) - g^*(t_n)) \geqslant 0.$$

Then

$$\begin{aligned} |q_n(t_n) - g^*(t_n)| &\geqslant |f(t_n) - u_n(t_n)| - |f(t_n) - g^*(t_n)| \\ &= d(f, G) - |f(t_n) - g^*(t_n)| \geqslant \delta. \end{aligned}$$

And

$$\begin{aligned} \|f - q_n\| &\geqslant |f(t_n) - q_n(t_n)| \\ &= |f(t_n) - u_n(t_n) + (1 - \lambda_n)(q_n(t_n) - g^*(t_n))| \\ &= |f(t_n) - u_n(t_n)| + (1 - \lambda_n)\, |q_n(t_n) - g^*(t_n)| \\ &\geqslant d(f, G) + (1 - \lambda_n)\delta. \end{aligned} \tag{13}$$

But for some $K > 0$, $\|q_n - g^*\| \leqslant K$. So

$$d(q_n, P_G(f)) \leqslant \|q_n - u_n\| = (1 - \lambda_n) \cdot \|q_n - g^*\| \leqslant K(1 - \lambda_n). \tag{14}$$

Equations (13) and (14) contradict (9). This proves that (i) implies (ii).

(ii) $\Rightarrow$ (i). Suppose

$$\|p - f\| \geqslant d(f, G) + s(f) \cdot d(p, P_G(f)), \qquad p \in G. \tag{15}$$

If (i) is false, then there exists $p \in G$ such that

$$\|p\|_V > 0, \tag{16}$$

$$\sup\{(f(t) - P_G(f))\, p(t) : t \in E(f - P_G(f))\} \leqslant 0. \tag{17}$$

From (17) we obtain that there exists an open set $W \supset E(f - P_G(f))$ such that

$$p(t) \cdot \operatorname{sign}(f(t) - g^*(t)) \leqslant -s(f)\, d(p, G(f))/2, \qquad t \in W. \tag{18}$$

Let $\varepsilon = d(f, G) - \max\{|f(t) - g^*(t)| : t \in T \setminus W\} > 0$. Then

$$|f(t) - g^*(t) - r \cdot p(t)| < d(f, G), \qquad 0 < r < \varepsilon/\|p\| \quad \text{and} \quad t \in T \setminus W. \tag{19}$$

Choose $t_r \in T$ such that

$$\|f-g^*-r\cdot p\| = |f(t_r)-g^*(t_r)-r\cdot p(t_r)|$$
$$= |\,|f(t_r)-g^*(t_r)| - r\cdot p(t_r)\cdot \operatorname{sign}(f(t_r)-g^*(t_r)-g^*(t_r))|. \tag{20}$$

If $0 < r < \varepsilon/\|p\|$, then (19) implies $t_r \in W$. It follows from (18) and (20) that

$$\|f-g^*-r\cdot p\| \leqslant d(f, G) + r\cdot s(f)\cdot d(p, G(f))/2. \tag{21}$$

But $d(g^* + r\cdot p, P_G(f)) = d(r\cdot p, P_G(f) - g^*) \geqslant d(r\cdot p, G(f)) = r\cdot d(p, G(f))$. This means that (21) contradicts (15).

(ii) $\Rightarrow$ (iii). Suppose

$$\|f-p\| \geqslant d(f, G) + r(f)\cdot d(p, P_G(f)), \qquad p \in G.$$

For any $p \in P_G(h)$, we have

$$\begin{aligned} r(f)\cdot d(p, P_G(f)) &\leqslant \|f-p\| - d(f, G) \\ &\leqslant \|f-h\| + \|h-p\| - d(f, G) \\ &= \|f-h\| + d(h, G) - d(f, G) \leqslant 2\,\|f-h\|. \end{aligned}$$

Hence

$$d(P_G(h), P_G(f)) \leqslant 2\cdot \|f-h\|/r(f), \qquad h \in C_0(T).$$

(iii) $\Rightarrow$ (i). Assume that statement (i) fails to be true; i.e., there exists $p \in G$ such that

$$\sup\{(f(t)-P_G(f,t))\,p(t) \colon t \in E(f-P_G(f))\} \leqslant 0, \qquad \|p\|_V \neq 0.$$

Define

$$f_\alpha(t) = [f(t)-g^*(t)-\alpha\cdot p(t)]_{-d(f,G)}^{d(f,G)} + g^*(t) + \alpha p(t),$$

where

$$[x]_a^b = \begin{cases} b, & x \geqslant b, \\ x, & a < x < b, \\ a, & x \leqslant a. \end{cases}$$

It is easy to check that

$$\begin{gathered} g^* + \alpha p \in P_G(f_\alpha), \\ \|f-f_\alpha\|/\alpha \to 0 \qquad \text{as} \quad \alpha \to 0^+. \end{gathered} \tag{22}$$

But

$$d(P_G(f_\alpha), P_G(f)) \geqslant d(g^* + \alpha p, P_G(f))$$
$$\geqslant d(\alpha p, G(f)) \geqslant \alpha \cdot \|p\|_V, \qquad \text{for} \quad \alpha > 0. \tag{23}$$

Equations (22) and (23) contradict the fact that P_G is upper Hausdorff–Lipschitz continuous at f.

(iii) $\Rightarrow$ (iv). Since P_G is upper Hausdorff–Lipschitz continuous at f, for any $p \in P_G(h)$ and $g \in P_G(f)$,

$$\|p - g\|_V \leqslant d(p, P_G(f)) \leqslant s(f) \cdot \|f - h\|. \tag{24}$$

By (5) and (24), we obtain

$$d(p - g, G(f)) \leqslant c \cdot \|p - g\|_V \leqslant c \cdot s(f) \cdot \|f - h\|. \tag{25}$$

For $g \in P_G(f)$, define

$$\delta(g) = d(f, G) - \max\{|f(t) - g(t)| : t \in T \setminus V\}.$$

Suppose $g \in P_G(f)$ such that

$$\delta(g) \geqslant (c \cdot s(f) + 2) \cdot \|f - h\|.$$

From (25) we obtain that for some $p \in P_G(h)$ and $q \in G(f)$,

$$\|g - p - q\| \leqslant c \cdot s(f) \cdot \|f - h\|.$$

For $t \in T \setminus V$,

$$|h(t) - p(t) - q(t)|$$
$$\leqslant |h(t) - f(t)| + |f(t) - g(t)| + |g(t) - p(t) + q(t)|$$
$$\leqslant \|f - h\| + d(f, G) - (c \cdot s(f) + 2)\, \|f - h\| + c \cdot s(f)\, \|f - h\|$$
$$\leqslant d(f, G) - \|f - h\| \leqslant d(h, G).$$

For $t \in V$,

$$|h(t) - p(t) - q(t)| = |h(t) - p(t)| \leqslant d(h, G).$$

Thus $p + q \in P_G(h)$ and

$$d(g, P_G(h)) \leqslant \|g - p - q\| \leqslant c \cdot s(f) \cdot \|f - h\|. \tag{26}$$

Set

$$\varepsilon = (c \cdot s(f) + 2)^{-1} \min\left\{\frac{\delta}{4}, d(f, G) - \frac{\delta}{2}\right\}.$$

Suppose

$$\|f-h\|<\varepsilon.$$

For $g \in P_G(f)$ with $\delta(g) \leqslant (c \cdot s(f)+2)\,\|f-h\|$, define

$$W=\left\{t \in T \setminus V : |f(t)-g(t)| \geqslant d(f, G)-\frac{\delta}{2}\right\}.$$

Since $|f(t)-g^*(t)| \leqslant d(f, G)-\delta$, we obtain

$$(f(t)-g(t))(g(t)-g^*(t)) \geqslant 0, \qquad t \in W, \tag{27}$$

$$|g(t)-g^*(t)| \geqslant \frac{\delta}{2}, \qquad t \in W. \tag{28}$$

Let

$$\lambda=(c \cdot s(f)+2)\,\|f-h\|/\|g-g^*\|.$$

Then from (27) and (28) we deduce

$$\begin{aligned}
&|f(t)-g(t)-\lambda(g(t)-g^*(t))| \\
&\quad=|f(t)-g(t)|-\lambda\,|g(t)-g^*(t)| \\
&\quad\leqslant d(f, G)-\lambda \cdot \delta/2 \\
&\quad\leqslant d(f, G)-(c \cdot s(f)+2) \cdot \|f-h\|, \qquad t \in W.
\end{aligned}$$

But for $t \in T \setminus (V \cup W)$, we have

$$\begin{aligned}
&|f(t)-g(t)-\lambda(g(t)-g^*(t))| \\
&\quad\leqslant d(f, G)-\frac{\delta}{2}+(c \cdot s(f)+2)\,\|f-h\| \\
&\quad\leqslant d(f, G)-(c \cdot s(f)+2) \cdot \|f-h\|.
\end{aligned}$$

Thus

$$\delta(g+\lambda(g-g^*)) \geqslant (c \cdot s(f)+2 \cdot \|f-h\|.$$

By (26) we get

$$d((g+\lambda(g-g^*)), P_G(h)) \leqslant c \cdot s(f) \cdot \|f-h\|.$$

And

$$\begin{aligned}
d(g, P_G(h)) &\leqslant c \cdot s(f) \cdot \|f-h\|+\|\lambda(g-g^*)\| \\
&\leqslant 2(c \cdot s(f)+1)\,\|f-h\|.
\end{aligned}$$

Hence

$$d(P_G(f), P_G(h)) \leqslant 2(c \cdot s(f) + 1 \,\|f-h\|,$$
$$h \in C_0(T) \quad \text{with } \|f-h\| < \varepsilon. \tag{29}$$

But (29) implies that for some $K > 0$ [2],

$$d(P_G(f), P_G(h)) \leqslant K \cdot \|f-h\|, \qquad h \in C_0(T).$$

Let $M = K + s(f)$. Then

$$D(P_G(f), P_G(h)) \leqslant M \cdot \|f-h\|, \qquad h \in C_0(T).$$

(iv) $\Rightarrow$ (iii). It is trivial.

The proof of Lemma 1 is completed now.

COROLLARY 1 [2]. *If $P_G(f)$ is unique, then the following are equivalent:*

(i) *$P_G(f)$ if strongly unique;*

(ii) *P_G is Hausdorff–Lipschitz continuous at f.*

Proof. Since $P_G(f)$ is unique, $P_G(f)$ is Hausdorff strongly unique if and only if $P_G(f)$ is trongly unique. Thus the corollary follows immediately from Lemma 1.

Lemma 1 can be considered as a generalization of Corollary 1 for multi-valued $P_G(f)$.

3. HAUSDORFF STRONG UNIQUENESS

In this section, we will show that if G satisfies condition (1), then $P_G(f)$ is Hausdorff strongly unique for every $f \in C_0(T)$.

From now on, we make use of the following notation:

$$G_B = \{g \in G \colon B \subset Z(g)\}, \qquad B \subset T;$$
$$Z(G_B) = \{t \in T \colon g(t) = 0 \text{ for all } g \in G_B\}.$$

LEMMA 2 [10]. *G satisfies condition* (1) *if and only if P_G is Hausdorff continuous at every $f \in C_0(T)$.*

LEMMA 3. *G satisfies condition* (1) *if and only if for any $\{t_i\}_0^r \subset T$ with*

$$\dim G\,|_{\{t_i\}_0^r} = \dim G\,|_{\{t_i\}_0^r \setminus \{t_j\}} = r, \qquad 0 \leqslant j \leqslant r, \tag{30}$$

there hold

$$\{t_i\}_0^r \subset \text{int } Z(G_{\{t_i\}_0^r}). \tag{31}$$

Proof. Necessity. It is an immediate corollary of Lemma 4 in [10] and Lemma 2.

Sufficiency. Assume that G does not satisfy (1), i.e., there exists nonzero $g \in G$ such that

$$\text{card}(bdZ(g)) \geqslant \dim G_{\text{int } Z(g)}. \tag{32}$$

From (32) we obtain that there exists $t_0 \in bdZ(g)$ such that

$$\dim G\,|_{Z(g)} = \dim G\,|_{Z(g) \setminus \{t_0\}}.$$

Select $t_1, ..., t_s \subset Z(g)$ such that

$$\dim G\,|_{\{t_i\}_0^s} = \dim G\,|_{\{t_i\}_0^s \setminus \{t_j\}} = s, \qquad 0 \leqslant j \leqslant s.$$

By the hypothesis of Lemma 3, we have

$$t_0 \subset \{t_i\}_0^s \subset \text{int } Z(G_{\{t_i\}_0^s}) \subset \text{int } Z(g).$$

This contradicts $t_0 \in bdZ(g)$.

LEMMA 4. *If $f \in C_0(T) \setminus G$ and $q \in P_G(f)$, then for any $p \in G$ with*

$$\text{int}\{t \in T: (f(t) - q(t))\, p(t) \geqslant 0\} \supset E(f - P_G(f)), \tag{33}$$

there hold

$$E(f - P_G(f)) \subset Z(G(f)) \subset Z(p), \tag{34}$$

where $G(f) = \text{span}\{g_1 - g_2 : g_1, g_2 \in P_G(f)\}$.

Proof. Let $g \in P_G(f)$ such that

$$E(f - g) = E(f - P_G(f)). \tag{35}$$

Since $E(f - P_G(f) \subset Z(g - q)$, we derive from (33) and (35) that there exists $\lambda > 0$ such that

$$g + \lambda p \in P_G(f).$$

Hence

$$E(f - P_G(f)) \subset Z(G(f)) \subset Z(\lambda p) = Z(p).$$

LEMMA 5. *Suppose that G satisfies condition* (1). *Then for any closed subset $Y \subset T$, $G^* = G|_Y$, and $h \in C_0(Y) \setminus G^*$, there hold*

$$E(h - P_{G^*}(h)) \subset \text{int } Z(G_{E(h - P_{G^*}(h))}). \tag{36}$$

Proof. *Let* $V = \text{int } Z(G_{E(h - P_{G^*}(h))})$. If (36) fails, then

$$A = E(h - P_{G^*}(h)) \setminus V \neq \phi. \tag{37}$$

If $\dim G_V|_A \leqslant \text{card}(A) - 1$, then there exist $t_0 \in A$ and $t_1, ..., t_r \in A \cup V$ such that

$$\dim G|_{\{t_i\}_0^r} = \dim G|_{\{t_i\}_0^r \setminus \{t_j\}} = r, \qquad 0 \leqslant j \leqslant r.$$

By Lemma 3 we obtain

$$\begin{aligned} t_0 \in \{t_i\}_0^r \subset \text{int } Z(G_{\{t_i\}_0^r}) \subset \text{int } Z(G_{A \cup V}) \\ = \text{int } Z(G_{E(h - P_{G^*}(h))}) = V. \end{aligned}$$

This is impossible.

If $\dim G_V|_A = \text{card}(A)$, set $g \in P_{G^*}(h)$; then there is $p \in G_V|_Y \subset G^*$ such that

$$p(t) = h(t) - g(t) \neq 0, \qquad t \in A. \tag{38}$$

Equations (37) and (38) imply

$$\text{int}_Y \{t \in Y: (h(t) - g(t))\, p(t) \geqslant 0\} \supset E(h - P_{G^*}(h)),$$

where $\text{int}_Y B$ denotes all interior points of B in Y. By Lemma 4, we have

$$E(h - P_{G^*}(h)) \subset Z(p).$$

This contradicts (38) and (37). The contradictions show that (36) is true.

LEMMA 6. *If G satisfies condition* (1), *then for any $f \in C_0(T)$ and $g \in P_G(f)$, set $E = E(f - P_G(f))$, $f - g|_E$ has zero as the unique best approximation from $G|_E$.*

Proof. We may assume $f \in C_0(T) \setminus G$. Let

$$h = f - g|_E, \qquad G^* = G|_E.$$

By the Kolmogorov criterion [13], we obtain

$$\|f - g\| = d(f, G) = d(h, G^*). \tag{39}$$

Lemma 5 states

$$E(h - P_{G^*}(h)) \subset \operatorname{int}(G_{E(h - P_{G^*}(h))}). \tag{40}$$

Let $p \in G$ such that

$$p\,|_E \in P_{G^*}(h), \tag{41}$$

$$\{t \in E\colon |h(t) - p(t)| = d(h, G^*)\} = E(h - P_{G^*}(h)). \tag{42}$$

From (39), (40), (41), and (42), we can deduce

$$\operatorname{int}\{t \in T\colon (f(t) - g(t))\,p(t) \geqslant 0\} \supset E = E(f - P_G(f)).$$

By Lemma 4, we get

$$E = E(f - P_G(f)) \subset Z(p). \tag{43}$$

And (39) and (43) imply $E = E(h - P_{G^*}(h))$. Hence, h has zero as the unique best approximation from G^*.

LEMMA 7. *If G satisfies condition* (1), *then $P_G(f)$ is Hausdorff strongly unique for all $f \in C_0(T)$.*

Proof. Obviously, $P_G(f)$ is strongly unique for all $f \in G$. Now suppose $f \in C_0(T) \setminus G$. Lemma 5 and 6 tell us that

$$E(f - P_G(f)) \subset \operatorname{int} Z(G_{E(f - P_G(f))}) = V.$$

This means that there exists $\alpha > 0$ such that

$$\|g\|_{E(f - P_G(f))} \leqslant \alpha\, \|g\|_V, \qquad g \in G. \tag{44}$$

By Lemma 6, we derive that there exists $\beta > 0$ such that

$$\max\{g(t)\operatorname{sign}(f(t) - P_G(f, t))\colon t \in E(f - P_G(f))\}$$
$$\geqslant \beta\, \|g\|_{E(f - P_G(f))}, \qquad g \in G. \tag{45}$$

By Lemma 2, we know that P_G is Hausdorff continuous at f. And (44), (45) imply that statement (i) in Lemma 1 holds for $r = \alpha \cdot \beta$. Thus $P_G(f)$ is Hausdorff strongly unique.

Remark. If $P_G(f)$ is strongly unique, then P_G is Hausdorff continuous at f. But, generally, the Hausdorff strong uniqueness of $P_G(f)$ does not imply that P_G is Hausdorff continuous at f.

4. CHARACTERIZATION OF $U_G = SU_G$

In this section, we will show that if T contains no isolated points, then $U_G = SU_G$ is equivalent to the fact that G satisfies condition (1). First we establish some more general results.

LEMMA 8. *If* $\dim G^* < \infty$, *then there exists a group of sets* $\{A_i\}_0^r \subset T$ *such that*

$$G_i|_{A_i} = G_i|_{A_i \setminus \{x\}} = \operatorname{card}(A_i) - 1 \geqslant 1, \qquad x \in A_i, \quad 0 \leqslant i \leqslant r, \tag{46}$$

$$\dim G_{r+1} = \operatorname{card}(T \setminus Z(G_{r+1})), \tag{47}$$

where $G_0 = G^*$ *and* $G_{i+1} = \{g \in G_i : A_i \subset Z(g)\}$, $0 \leqslant i \leqslant r$.

Proof. This lemma can be easily proved by induction.

LEMMA 9. *Suppose* $f \in C_0(T) \setminus G$ and $g \in P_G(f)$ *such that*

$$E(f-g) = E(f - P_G(f)). \tag{48}$$

If $h \in C_0(T)$ *satisfies*

$$\|h\| = d(f, G) = \|f - g\|, \tag{49}$$

$$\operatorname{int}\{t \in T : h(t) = f(t) - g(t)\} \supset E(f - P_G(f)), \tag{50}$$

then

$$Z(G(f)) \subset Z(G(h)) = Z(P_G(h)). \tag{51}$$

Proof. From (49), (50), we obtain that $0 \in P_G(h)$ and $d(h, G) = d(f, G)$. If $p \in P_G(h)$, then for all $0 \leqslant \lambda \leqslant 1$, $\lambda p \in P_G(h)$. Let

$$V = \operatorname{int}\{t \in T : h(t) = f(t) - g(t)\}.$$

Then for $0 < \lambda < 1$,

$$\begin{aligned} |f(t) - g(t) - \lambda p(t)| &= |h(t) - \lambda p(t)| \\ &\leqslant d(h, G) = d(f, G) = \|f - g\|, \qquad t \in V. \end{aligned} \tag{52}$$

By (48) and (50), we obtain that for some $0 < \lambda^* < 1$,

$$|f(t) - g(t) - \lambda^* p(t)| \leqslant \|f - g\|, \qquad t \in T \setminus V. \tag{53}$$

Equations (52) and (53) mean $g + \lambda^* p \in P_G(f)$. So

$$Z(G(f)) \subset Z(g + \lambda^* p - g) = Z(p), \qquad p \in P_G(h).$$

This implies that (51) holds.

LEMMA 10. *If there exist* $f \in C_0(T)\backslash G$ *and* $g \in G$ *such that*

$$Z(G(f))\backslash Z(g) \neq \phi, \tag{54}$$

$$\max\{g(t)\operatorname{sign}(f(t)-P_G(f,t)) : t \in E(f-P_G(f))\} \leqslant 0, \tag{55}$$

then there exist $h \in C_0(T)\backslash G$ and $p \in G$ such that

$$Z(G(h))\backslash Z(p) \neq \phi, \tag{56}$$

$$\max\{p(t)\operatorname{sign}(h(t)-P_G(h,t)) : t \in E(h-P_G(h))\} \leqslant 0, \tag{57}$$

$$\dim G(h) = \operatorname{card}(T\backslash Z(G(h))). \tag{58}$$

Proof. Let $q \in P_G(f)$ such that

$$E(f-q) = E(f-P_G(f)). \tag{59}$$

Set $G^* = G_{Z(G(f))}$. From Lemma 8, we obtain that there is a group of sets $\{A_i\}_0^r$ satisfying (46) and (47). Arbitrarily choose $t_i \in A_i$, $0 \leqslant i \leqslant r$. From (46) we know that there is $g^* \in G^*$ such that

$$\left(\bigcup_{i=0}^{r} A_i\right)\Big\backslash \{t_i\}_0^r \subset Z(g-g^*).$$

There are $\varepsilon_i \in \{-1, 1\}$, $0 \leqslant i \leqslant r$, such that

$$\varepsilon_i(g(t_i)-g^*(t_i)) \leqslant 0.$$

Equation (46) also implies that there exist extremal signatures [12] σ_i of G_i supporting on A_i such that

$$\sigma_i(t_i) = \varepsilon_i, \qquad 0 \leqslant i \leqslant r.$$

Then

$$\sigma_i(t)\cdot(g(t)-g^*(t)) \leqslant 0, \qquad t \in A_i, \quad 0 \leqslant i \leqslant r. \tag{60}$$

By Tietz's extension theorem and (55), (60), we can construct $h \in C_0(T)$ satisfying

$$V = \operatorname{int}\{t \in T : h(t) = f(t) - q(t)\} \supset E(f-P_G(f)) = E(f-q);$$

$$\|f-q\| = d(f; G) = \|h\|; \tag{61}$$

$$h(t) = \sigma_i(t), \qquad t \in A_i, \quad 0 \leqslant i \leqslant r; \tag{62}$$

$$\max\{(g(t)-g^*(t))\operatorname{sign} h(t) : t \in E(h)\} \leqslant 0. \tag{63}$$

It follows from Lemma 9 that $Z(G(f)) \subset Z(P_G(h))$. So

$$P_G(h) \subset G_{Z(G(f))} = G^*. \tag{64}$$

Since σ_i are extremal signatures of G_i, $0 \leqslant i \leqslant r$, by (61), (62), and (64), we obtain

$$\bigcup_{i=0}^{r} A_i = \bigcup_{i=0}^{r} \sup \sigma_i \subset Z(P_G(h)). \tag{65}$$

Equations (64) and (65) imply

$$G(h) = \text{span } P_G(h) \subset G_{r+1}.$$

It follows from (47) that

$$\dim G(h) = \text{card}(T \setminus Z(G(h))). \tag{66}$$

Let $p = g - g^*$. Then, by (54), (63), (64), and $g^* \in G_{Z(G(f))}$, we obtain that

$$Z(G(h)) \setminus Z(p) \supset Z(G(f)) \setminus Z(p) = Z(G(f)) \setminus Z(g) \neq \phi, \tag{67}$$

$$\max\{p(t) \text{ sign}(h(t)) - P_G(h; t)): t \in E(h - P_G(h))\} \leqslant 0. \tag{68}$$

Equations (66), (67), and (68) complete the proof of this lemma.

LEMMA 11. *Suppose that G^* is a finite-dimensional subspace of $C_0(T)$. If $z \in bdZ(G^*)$, $z_k \in T \setminus Z(G^*)$, and $z_k \to z$, then there exist $\lambda_k > 0$ and $\lambda_k \to 0$ such that*

$$\limsup_{k \to \infty} |g(z_k)|/\lambda_k = +\infty, \qquad \textit{for all } g \in G^* \quad \textit{with} \quad Z(g) \cap \{z_k\}_1^\infty = \phi. \tag{69}$$

Proof. Assume that no $\{\lambda_k\}$ satisfies (69). Then there are $\{g_n\}_0^\infty \subset G^*$ and $M_n > 0$ such that

$$Z(g_n) \cap \{z_k\}_1^\infty = \phi, \tag{70}$$

$$|g_n(z_k)| \leqslant M_n \cdot |g_{n-1}(z_k)|^2, \qquad k = 1, 2, ..., n = 1, 2, ... \tag{71}$$

From (70) and (71) we deduce that $\{g_n\}_0^\infty$ is a linearly independent system in G^*. This contradicts that dim G^* is finite.

LEMMA 12. *If G does not satisfy condition (1), then there exist $f \in C_0(T) \setminus G$ and $g \in G$ such that*

$$Z(G(f)) \setminus Z(g) \neq \phi, \tag{72}$$

$$\max\{g(t) \text{ sign}(f(t) - P_G(f, t)): t \in E(f - P_G(f))\} \leqslant 0. \tag{73}$$

Proof. By Lemma 3, there are $\{t_i\}_0^r$ such that

$$\dim G\,|_{\{t_i\}_0^r} = \dim G\,|_{\{t_i\}_0^r \setminus \{t_j\}} = r, \qquad 0 \leqslant \delta \leqslant r, \tag{74}$$

$$t_r \bar{\in} \operatorname{int} Z(G_{\{t_i\}_0^r}). \tag{75}$$

Obviously, there exists an open neighborhood V of t_r such that for any $g \in G$, $t_r \in \operatorname{int} Z(g)$ implies $V \subset Z(g)$. Set $G^* = G_{\{t_i\}_0^r}$. Let $t_{r+1}, ..., t_n \in V$ such that

$$\dim G^*\,|_{\{t_i\}_{r+1}^n} = \dim G^*\,|_V = n - r.$$

Select $z_k \in V \setminus Z(G^*)$ such that $z_k \to t_r$ as $k \to \infty$. By selecting a subsequence, we may assume that there exist $r+1 \leqslant m \leqslant n$ and $\varepsilon_i \in \{-1, 1\}$, $r \leqslant i \leqslant m$, such that

$$\dim G^*\,|_{\{t_i\}_{r+1}^m \cup \{z_k\}} = \dim G^*\,|_{\{t_i\}_{r+1}^m} = \dim G^*\,|_{\{t_i\}_{r+1}^m \cup \{z_k\} \setminus \{t_j\}} = m - r,$$
$$r+1 \leqslant j \leqslant m, \quad k \geqslant 1, \tag{76}$$

and

$$\sigma_k(t) = \begin{cases} \varepsilon_i, & t = t_i, \quad r+1 \leqslant i \leqslant m, \\ \varepsilon_r, & t = z_k, \\ 0 & \text{otherwise}, \end{cases}$$

are extremal signatures of G^*. Let $g^* \in G^*$ satisfy

$$g^*(t_i) = -\varepsilon_i, \qquad r+1 \leqslant i \leqslant m.$$

Let V_i be open neighborhoods of t_i such that

$$\varepsilon_i g^*(t) \leqslant 0, \qquad t \in V_i, \quad r+1 \leqslant i \leqslant m. \tag{77}$$

By Lemma 11, there are $0 < \lambda_k < 1$ such that

$$\limsup_{k \to \infty} |g(z_k)|/\lambda_k = +\infty, \qquad \text{for} \quad g \in G^* \quad \text{with} \quad Z(g) \cap \{z_k\}_1^\infty = \phi.$$

Equation (74) implies that there is an extremal signature σ of G supporting on $\{t_i\}_0^r$ such that $\sigma(t_r) = \varepsilon_r$. By Tietz's extension theorem, we can construct $f \in C_0(T)$ satisfying

$$f(t_i) = \begin{cases} \sigma(t_i), & 0 \leqslant i \leqslant r, \\ \varepsilon_i, & r+1 \leqslant i \leqslant m, \end{cases} \tag{78}$$

$$1 > f(z_k) \cdot \sigma(t_r) \geqslant (1 - \lambda_k), \qquad k \geqslant 1,$$

$$\|f\| = 1, \tag{79}$$

$$E(f) \subset Z(G^*) \cup \left(\bigcup_{i=r+1}^{m} V_i \right), \tag{80}$$

$$\varepsilon_i f(t) \geqslant 0, \qquad t \in V_i, \quad r+1 \leqslant i \leqslant m. \tag{81}$$

We first show that

$$\{t_i\}_0^m \cup \{z_k\}_1^\infty \subset Z(G(f)). \tag{82}$$

In fact, (78), (79), and σ being an extremal signature of G imply that $0 \in P_G(f)$ and

$$\{t_i\}_0^r \subset Z(G(f)) = Z(P_G(f)). \tag{83}$$

Let $p \in P_G(f) \subset G^*$. Set $B = \{k: p(z_k) \cdot \sigma(t_r) \geqslant 0\}$. If $B = \phi$, by the property of $\{\lambda_k\}_1^\infty$, there is some k such that

$$-\sigma(t_r)\, P(z_k) \geqslant 2\lambda_k.$$

Hence,

$$\begin{aligned} |f(z_k) - p(z_k)| &= \sigma(t_r) f(z_k) - \sigma(t_r)\, p(z_k) \geqslant 1 - \lambda_k - \sigma(t_r)\, p(z_k) \\ &\geqslant 1 - \lambda_k + 2\lambda_k = 1 + \lambda_k > 1 = d(f, G) = \|f - p\|. \end{aligned}$$

This is impossible.

Now arbitrarily choose $k \in B$. By (78) and the definition of σ_k, we have

$$\begin{aligned} p(t_i)\sigma_k(t_i) &= p(t_i)\varepsilon_i \geqslant 0, \qquad r+1 \leqslant i \leqslant m, \\ p(z_k)\sigma_k(z_k) &= p(z_k)\sigma(t_r) \geqslant 0. \end{aligned}$$

Since σ_k is an extremal signature of G^*, we obtain

$$\{t_i\}_{r+1}^m \cup \{z_k\} \subset Z(p), \qquad p \in P_G(f).$$

This and (76), (83) mean that (82) is true.

From (77), (80), and (81), we obtain

$$\begin{aligned} &\max\{g^*(t)\,\mathrm{sign}(f(t) - P_G(f, t)) : t \in E(f - P_G(f))\} \\ &\quad \leqslant \max\{g^*(t) \cdot \mathrm{sign}\, f(t) : t \in E(f)\} \leqslant 0. \end{aligned} \tag{84}$$

And

$$Z(G(f)) \setminus Z(g^*) \supset \{t_i\}_{r+1}^m \neq \phi. \tag{85}$$

Equations (84) and (85) are the required results.

LEMMA 13. *If the Hausdorff continuity of P_G at f always implies the Hausdorff–Lipschitz continuity of P_G at f, then G satisfies condition* (1).

Proof. Assume that G does not satisfy condition (1). Then Lemma 12 tells us that there are $f \in C_0(T) \backslash G$, $g \in G$ satisfying (72) and (73). By Lemma 10, we can find $h \in C_0(T) \backslash G$ and $p \in G$ such that (56), (57), and (58) hold. But (58) means that $Z(G(h))$ is an open subset. It follows from Theorem B that P_G is Hausdorff continuous at f. By the hypothesis of this lemma, we conclude that P_G is Hausdorff–Lipschitz continuous at f. It is derived from Lemma 1 that

$$\max\{p(t)\operatorname{sign}(h(t) - P_G(h, t)) : t \in E(h - P_G(h))\} \geqslant r \cdot \|p\|_{Z(G(h))}.$$

This contradicts (56) and (57). The contradiction shows that G satisfies condition (1).

If T has no isolated points, then (58) implies that $P_G(h)$ is unique. If $U_G = SU_G$, then Lemma 1 also ensures (84) which contradicts (56) and (57). Thus, we have the following characteristic description of $U_G = SU_G$:

LEMMA 14. *Suppose that T has no isolated points. Then G satisfies condition* (1) *if and only if $U_G = SU_G$.*

Proof. This lemma follows immediately from Lemma 7 and the remark above.

5. SUMMARY OF PROVED RESULTS AND SOME REMARKS

First we summarize the results proved in Section 2, 3, and 4.

PROPOSITION 1. *The following are equivalent*:

(i) *G satisfies condition* (1);

(ii) *P_G is Hausdorff continuous at every $f \in C_0(T)$*;

(iii) *P_G is upper Hausdorff–Lipschitz continuous at every $f \in C_0(T)$*;

(iv) *P_G is Hausdorff–Lipschitz continuous at every $f \in C_0(T)$*;

(v) *$P_G(f)$ is Hausdorff strongly unique for all $f \in C_0(T)$*;

(vi) *Hausdorff continuity of P_G at f always implies Hausdorff–Lipschitz continuity of P_G at f*;

(vii) *Hausdorff continuity of P_G at f always implies upper Hausdorff–Lipschitz continuity of P_G at f*;

(viii) *Hausdorff continuity of P_G at f always implies Hausdorff strong uniqueness of $P_G(f)$.*

If T has no isolated points, then all above are equivalent to

(ix) $U_G = SU_G$.

Furthermore, if T is connected, then all above are equivalent to

(x) *G satisfies the Haar condition.*

Proof. The equivalences among (i)–(ix) follow from Lemma 1, Lemma 2, Lemma 7, Lemma 13, and Lemma 14. Under the hypothesis that T is connected, Blatter *et al.* [4] show that P_G is Hausdorff continuous at every $f \in C_0(T)$ if and only if G satisfies the Haar condition. Thus (ii) implies (x). This completes the proof of Proposition 1.

Theorem 1 is only a part of Proposition 1.

Remark. Recall that G is an almost Chebyshev subspace of $C_0(T)$ if except for a set of first category in $C_0(T)$ every function has a unique best approximation from G [8]. There are nice characterizations about almost Chebyshev subspaces:

THEOREM C. *Suppose that T is a compact metric space and $G \subset C_0(T)$ with* $\dim G < \infty$. *Then the following are equivalent:*

(i) *G is an almost Chebyshev subspace;*

(ii) *if $V \subset T$ is open and* $\dim G_V \geqslant 1$, *then* $\mathrm{card}(V) = \dim G|_V$;

(iii) *SU_G is dense in $C_0(T)$;*

(iv) *if P_G is Hausdorff continuous at f, then $P_G(f)$ is unique.*

The equivalence of (i) and (ii) is proved by Garkavi [8]; Nürnberger and Singer show the equivalence of (i) and (iii) [14]; Bartelt and Schmidt [2] establish the equivalence of (i) and (iv).

Actually, if T is a locally compact Hausdorff space, then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) (see [2, 8, 14]). On the other hand, if G satisfies (ii) in Theorem C, then (58) implies that $P_G(h)$ is unique. Thus, we have the following corollary of Proposition 1 and Theorem C:

COROLLARY 2. *If G is a finite-dimensional almost Chebyshev subspace of $C_0(T)$, then the statements* (i)–(x) *in Proposition* 1 *are mutually equivalent.*

We leave the details to the interested reader.

Remark. Generally, if T contains isolated points, the equivalence of (i) and (v) in Theorem 1 may not be true. For example, let G_0 be any finite-dimensional subspace of $C[0, 1]$. Let $T = [0, 1] \cup \{-1\}$. Define

$$G = \{g \in C(T) = g \mid [0, 1] \in G_0\}.$$

Then $\dim G = \dim G_0 + 1$. For any $f \in C(T)$, $f \in U_G$ if and only if $f \in G$. So $U_G = SU_G$. But it is easy to check that G satisfies (i) if and only if G_0 satisfies the Haar condition. Thus (i) and (v) in Theorem 1 are not equivalent if G_0 is not a Haar subspace of $C[0, 1]$.

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